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JOURNAL OF DIFFERENTIAL EQUATIONS 95, 1–19 (1992)

Properties of Potential Systems in $\mathbf{R}^N$

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Received January 4, 1990

1. INTRODUCTION

Our purpose is to obtain existence theorems and asymptotic properties as $|x| \to \infty$ for potential systems of the type

$$-\Delta u^i + a^i(x)\, u^i = \lambda f^i(x, \mathbf{u}), \qquad x \in \mathbf{R}^N, \qquad N \geqslant 3, \qquad i = 1, ..., M, \quad (1)$$

where $x = (x^1, ..., x^N)$, $\mathbf{u} = (u^1, ..., u^M)$, Δ denotes the N-dimensional Laplacian, λ is a positive parameter, $a^i(x)$ is nonnegative and bounded, $a^i \in C^\eta_{\mathrm{loc}}(\mathbf{R}^N)$, $f^i \in C^\eta_{\mathrm{loc}}(\mathbf{R}^N \times \mathbf{R}^M, \mathbf{R})$ for some fixed $\eta \in (0, 1)$, $f^i(x, \mathbf{u}) > 0$ if every $u^j > 0$, $j = 1, ..., M$, and f^i satisfies additional conditions to be listed in Sections 2 and 3, $i = 1, ..., M$. The definition of a *potential system* (1) is that there exists a differentiable function F with respect to $u^1, ..., u^M$ such that

$$\frac{\partial}{\partial u^i} F(x, u^1, ..., u^M) = f^i(x, u^1, ..., u^M), \qquad i = 1, ..., M. \qquad (2)$$

In vector-matrix notation, the system (1) is

$$-\Delta \mathbf{u} + A\mathbf{u} = \lambda f(x, \mathbf{u}), \qquad x \in \mathbf{R}^N, \qquad N \geqslant 3, \qquad (1')$$

where A denotes the diagonal matrix with diagonal entries $a^1, ..., a^M$. The usual vector length is $|\mathbf{u}| = [\sum_{i=1}^M |u^i|^2]^{1/2}$ and we also use the notation $\nabla\mathbf{u} \cdot \nabla\mathbf{v} = \sum_{i=1}^M \nabla u^i \cdot \nabla v^i$, $|\nabla \mathbf{u}|^2 = \nabla\mathbf{u} \cdot \nabla\mathbf{u}$. Vector inequalities $\mathbf{u} > \mathbf{v}$ are understood to mean $u^i > v^i$ for each $i = 1, ..., M$.

The letter C will be used from time to time as a generic notation for a

* Supported by the Australian Research Council.

† Supported by NSERC (Canada) under Grant 5-83105.

0022-0396/92 $3.00

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positive constant, whose value is inconsequential and may change from one place to another. The dependence of C on parameters will be specified only when significant. The open ball in $\mathbf{R}^N$ of radius r and centre x will be denoted by $B_r(x)$.

We use the notation $\| \|_q$, $\| \|_{q,B}$ for the norms in $L^q(\mathbf{R}^N)$, $L^q(B)$, respectively, for $B \subset \mathbf{R}^N$. The standard Sobolev spaces $W_0^{1,2}(\mathbf{R}^N)$ and $D_0^{1,2}(\mathbf{R}^N)$ are the completions of $C_0^\infty(\mathbf{R}^N)$ in the respective norms

$$\|\phi\|_{1,2} = [\|\nabla\phi\|_2^2 + \|\phi\|_2^2]^{1/2} \qquad \text{and} \qquad \|\phi\| = \|\nabla\phi\|_2.$$

We also employ Sobolev spaces of vector functions $\mathbf{u}: \mathbf{R}^M \to \mathbf{R}$, defined by

$$E = \{\mathbf{u} : u^i \in W_0^{1,2}(\mathbf{R}^N), i = 1, ..., M\}$$

and

$$D = \{\mathbf{u} : u^i \in D_0^{1,2}(\mathbf{R}^N), i = 1, ..., M\}$$

with respective norms

$$\|\mathbf{u}\|_E = [\| \, |\nabla \mathbf{u}| \, \|_2^2 + \| \, |\mathbf{u}| \, \|_2^2]^{1/2} = \left[\sum_{i=1}^{M} \|u^i\|_{1,2}^2 \right]^{1/2}$$

and

$$\|\mathbf{u}\|_D = \| \, |\nabla \mathbf{u}| \, \|_2.$$

The existence of a positive solution pair λ, u of (1) will be proved in two cases (i) $a^i(x)$ is bounded below by a positive constant, and (ii) $a^i(x) \geqslant 0$ in $\mathbf{R}^N$, $i = 1, ..., M$. In both cases $f^i(x, \mathbf{u})$ is assumed to satisfy a power growth condition in the components of $\mathbf{u}$, with exponents $\beta = (\beta^1, ..., \beta^M)$ of each power of $\mathbf{u}$ restricted by $1 < |\beta| < (N+2)/(N-2)$ (in multi-index notation; see Section 2). Also in both cases $f^i(x, \mathbf{u}) \to 0$ as $|x| \to \infty$, uniformly in $\mathbf{u}$, but the decay law in case (ii) must be more specific, as described in (A_6) (see Section 3). The necessity of such conditions for the existence of positive solutions $\mathbf{u} \in D$ of (1) is indicated in Section 5. None of our results require (1) to be radially symmetric.

In case (i) a direct variational approach is used to construct a positive solution $\mathbf{u} \in E$ (Theorem 1), but this is not possible in case (ii) since (1) generally has no $L^2(\mathbf{R}^N)$-solutions; this is well known if $M = 1$, $a^1 \equiv 0$, for example. Instead Theorem 5 yields positive solutions in D (so in $L^{2N/(N-2)}(\mathbf{R}^N)$) by consideration of a sequence of Eq. $(16)_n$, $n = 1, 2, ...$, of type (i) and letting $n \to \infty$.

A priori asymptotic properties of nonnegative solutions $\mathbf{u} \in D$ of (1) are proved in Section 4. In particular, conditions on f are given which guarantee that every such solution satisfies $u^i(x) = 0(|x|^{2-N})$ as $|x| \to \infty$, $i = 1, ..., M$. Similar results have been obtained recently by Egnell [12], Li and Ni [22], and Pucci and Serrin [34] for scalar equations, and sometimes restricted to radially symmetric equations. Of the many recent investigations of problems (1) having $M=1$ we mention only [2–6, 8–16, 18–28, 30–32, 36–38, 41]. The few available results for elliptic systems in $\mathbf{R}^N$ [1, 9, 19, 23, 33] are not of our type here.

2. Existence of Solutions in the Case $\mathbf{a}(x) \geqslant a_* > 0$

The existence of positive solutions u of (1) will first be proved under the assumption $a^i(x) \geqslant a_* > 0$ for all $x \in \mathbf{R}^N$, $i=1, ..., M$. The assumptions for $f^i(x, \mathbf{u})$ are listed below. We use the notation

$$f^{ij}(x, \mathbf{u}) = \frac{\partial}{\partial u^j} f^i(x, \mathbf{u}), \qquad i, j = 1, ..., M,$$

and for a multi-index $\alpha = (\alpha^1, ..., \alpha^M) \geqslant 0$,

$$\mathbf{u}^\alpha = \prod_{i=1}^M (u^i)^{\alpha^i} \quad \text{if} \quad \mathbf{u} \geqslant 0, \qquad |\alpha| = \sum_{i=1}^M \alpha^i,$$

$$|\mathbf{u}^\alpha| = \prod_{i=1}^M |u^i|^{\alpha^i} \quad \text{if} \quad \mathbf{u} \in \mathbf{R}^M.$$

Assumptions for $\mathbf{f}(x, \mathbf{u})$

(A_1) There exists a nontrivial potential function F satisfying (2) such that $F \in C^{1+\eta}_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}^M, \mathbf{R}_+)$, where $\mathbf{R}_+ = [0, \infty)$, $F(x, \mathbf{u})$ is twice continuously differentiable with respect to the components of $\mathbf{u}$, $F(x, \mathbf{u}) = 0$ if any component $u^i = 0$, and $F(x, -\mathbf{u}) = F(x, \mathbf{u})$ for all x, $\mathbf{u}$.

(A_2) There exists a continuous function $g\colon \mathbf{R}^N \to \mathbf{R}_+$ with $\lim_{|x| \to \infty} g(x) = 0$ such that

$$F(x, \mathbf{u}) \leqslant g(x) \sum_{k=1}^K \mathbf{u}^{\alpha_k}, \quad x \in \mathbf{R}^N, \qquad \mathbf{u} \in \mathbf{R}^M_+$$

for some multi-indices α_k with $2 < |\alpha_k| < 2N/(N-2)$, $k = 1, ..., K$.

(A_3) $f^i(x, \mathbf{u}) > 0$ for all $\mathbf{u} > 0$, $x \in \mathbf{R}^N$ and

$$|f^i(x, \mathbf{u})| \leqslant Cg(x) \sum_{k=1}^K |\mathbf{u}^{\beta_{ik}}|, \qquad x \in \mathbf{R}^N, \qquad \mathbf{u} \in \mathbf{R}^M$$

for some constant C and multi-indices β_{ik} with

$$1 < |\beta_{ik}| < (N+2)/(N-2), \qquad i = 1, ..., M, \qquad k = 1, ..., K.$$

(A_4) $|f^{ij}(x, \mathbf{u})| \leqslant Cg(x) \sum_{k=1}^{K} |\mathbf{u}^{\gamma_{ijk}}|$, $x \in \mathbf{R}^N$, $\mathbf{u} \in \mathbf{R}^M$ for some constant C and multi-indices γ_{ijk} satisfying

$$|\gamma_{ijk}| < 4/(N-2), \qquad i, j = 1, ..., M, \qquad k = 1, ..., K.$$

For example, all these conditions hold for systems (1) with potential of the form

$$F(x, \mathbf{u}) = \sum_{k=1}^{K} g^k(x)\, |\mathbf{u}^{\alpha_k}|, \qquad x \in \mathbf{R}^N, \qquad 3 \leqslant N \leqslant 5, \tag{3}$$

where each $g^k \in C^{1+\eta}(\mathbf{R}^N)$, $0 \not\equiv g^k(x) \geqslant 0$, $\lim_{|x| \to \infty} g^k(x) = 0$, either $\alpha_k^h = 1$ or $\alpha_k^h > 2$ for each $h = 1, ..., M$, $k = 1, ..., K$, with $\alpha_k^h > 2$ for at least one component of each multi-index α_k, and $|\alpha_k| < 2N/(N-2)$, $k = 1, ..., K$. We note that these restrictions on the multi-indices place serious limitations on the problems (1) which can be considered: $M \leqslant 4$ if $N = 3$; $M \leqslant 2$ if $N = 4$ or $N = 5$. Such restrictions seem essential for the existence of positive solutions $\mathbf{u}$ of (1) in $\mathbf{R}^N$ in view of the necessary conditions in Section 5. If the problem is weakened to seeking nontrivial nonnegative solutions, the restrictions on the multi-indices in (3) can be considerably weakened, e.g., some components of α_k can be zero.

THEOREM 1. *If* $a^i \geqslant a_* > 0$, $i = 1, ..., M$ *and* (A_1)–(A_4) *hold, the system* (1) *has a positive solution pair* λ, $\mathbf{u}$ *with* $\mathbf{u} \in E \cap C^{2+\eta}_{\text{loc}}(\mathbf{R}^N)$.

The first step in the proof is to construct a weak solution of (1) by applying the methods of the Calculus of Variations to the constrained minimization problem

$$I^* = \inf\{I(\mathbf{u}) : \mathbf{u} \in E,\ J(\mathbf{u}) = d > 0\}, \tag{4}$$

where the functionals I and J are defined by

$$I(\mathbf{u}) = \frac{1}{2} \int_{\mathbf{R}^N} (|\nabla \mathbf{u}|^2 + A\mathbf{u} \cdot \mathbf{u})\, dx, \qquad \mathbf{u} \in E \tag{5}$$

$$J(\mathbf{u}) = \int_{\mathbf{R}^N} F(x, \mathbf{u})\, dx, \qquad \mathbf{u} \in E. \tag{6}$$

Standard procedure shows that I is well defined and continuously Fréchet differentiable on E, with Fréchet derivative $I'(u)$ given by

$$I'(\mathbf{u})\,\mathbf{v} = \int_{\mathbf{R}^N} (\nabla\mathbf{u}\cdot\nabla\mathbf{v} + A\mathbf{u}\cdot\mathbf{v})\,dx \tag{7}$$

for all $\mathbf{u}, \mathbf{v} \in E$. An analogue of this for J will now be proved.

LEMMA 2. *The functional J is well defined and of class $C^1(E)$ with Fréchet derivative given by*

$$J'(\mathbf{u})\,\mathbf{v} = \int_{\mathbf{R}^N} \mathbf{f}(x, \mathbf{u})\cdot\mathbf{v}\,dx \tag{8}$$

for all $\mathbf{u}, \mathbf{v} \in E$, where $\mathbf{f} = (f^1, ..., f^M)$.

Proof. For $\alpha = (\alpha^1, ..., \alpha^M) > 0$ and $u^h \in L^{|\alpha|}(\mathbf{R}^N)$ for $h = 1, ..., M$, repeated use of Hölder's inequality shows that

$$\left|\int_{\mathbf{R}^N} \mathbf{u}^\alpha(x)\,dx\right| \leqslant \prod_{h=1}^{M} \|u^h\|_{|\alpha|}^{\alpha^h}. \tag{9}$$

It follows from (6), (9), and assumption (A_2) that

$$J(\mathbf{u}) \leqslant C \sum_{k=1}^{K} \prod_{h=1}^{M} \|u^h\|_{|\alpha_k|}^{\alpha_k^h}$$

for some positive constant C. Since $|\alpha_k| < 2N/(N-2)$ for $k = 1, ..., K$, the Sobolev embedding $W_0^{1,2}(\mathbf{R}^N) \hookrightarrow L^{|\alpha_k|}(\mathbf{R}^N)$ shows that $J(\mathbf{u})$ is well defined on E.

To prove (8) we note from Taylor's formula for $\phi(t) = F(x, \mathbf{u} + t\mathbf{v})$, $\mathbf{u}, \mathbf{v} \in E$, and (A_4) that

$$\begin{aligned}
&\left|\frac{J(\mathbf{u}+t\mathbf{v}) - J(\mathbf{u})}{t} - \int_{\mathbf{R}^N} f(x, \mathbf{u})\cdot\mathbf{v}\,dx\right| \\
&\quad = \left|\frac{1}{t}\int_{\mathbf{R}^N} \Big[F(x, \mathbf{u}+t\mathbf{v}) - F(x, \mathbf{u}) - t\sum_{i=1}^{M} f^i(x, \mathbf{u})\,v^i\Big]\,dx\right| \\
&\quad = \left|\frac{t}{2}\int_{\mathbf{R}^N} \sum_{i,j=1}^{M} f^{ij}(x, \boldsymbol{u}+\theta t\mathbf{v})\,v^i v^j\,dx\right| \\
&\quad \leqslant C\,|t| \int_{\mathbf{R}^N} \sum_{i,j=1}^{M} \sum_{k=1}^{K} |(\mathbf{u}+\theta t\mathbf{v})^{\gamma_{ijk}}|\,|v^i|\,|v^j|\,dx
\end{aligned}$$

for some $\theta=\theta(x)\in(0,1)$ and some positive constant C. We now use (9) with the choice $\alpha=(\gamma_{ijk}^1, ..., \gamma_{ijk}^M, 1, 1)$. By (A_4), $|\alpha|=|\gamma_{ijk}|+2<2N/(N-2)$ and hence the embedding $W_0^{1,2}(\mathbf{R}^N) \hookrightarrow L^{|\alpha|}(\mathbf{R}^N)$ shows that $u^i, v^i \in L^{|\alpha|}(\mathbf{R}^N)$ for $i=1, ..., M$ and $\|u^h+\theta t v^h\|_{|\alpha|}$ is a bounded function of t for $|t|\leqslant 1$. Use of (9) for this α then implies that

$$\left|\frac{J(\mathbf{u}+t\mathbf{v})-J(\mathbf{u})}{t}-\int_{\mathbf{R}^N}\mathbf{f}(x,\mathbf{u})\cdot\mathbf{v}\,dx\right|$$

$$\leqslant C\,|t| \sum_{i,j=1}^{M}\sum_{k=1}^{K}\prod_{h=1}^{M}\|u^h+\theta t v^h\|_{|\alpha|}^{\gamma_{ijk}^h}\,\|v^i\|_{|\alpha|}\,\|v^j\|_{|\alpha|}$$

$$\to 0 \qquad \text{as} \qquad t\to 0, \tag{10}$$

thereby establishing (8).

To prove that $J\in C^1(E)$, let $\mathbf{u}, \mathbf{v}\in E$ and let $\{\mathbf{u}_n\}$ be a sequence in E with limit $\mathbf{u}$ in the E-topology. Let α be the same multi-index as in (10), so $|\alpha|<2N/(N-2)$. Then the embedding $W_0^{1,2}(\mathbf{R}^N) \hookrightarrow L^{|\alpha|}(\mathbf{R}^N)$ implies that there exists a constant C_0, independent of n, such that

$$\|u_n^j-u^j\|_{|\alpha|}\leqslant C_0\,\|u_n^j-u^j\|_{1,2}\to 0 \qquad \text{as} \qquad n\to\infty$$

for $j=1, ..., M$. It follows from the mean value theorem, (A_4), and (9) by the procedure used for (10) that

$$\left|\int_{\mathbf{R}^N}[\mathbf{f}(x,\mathbf{u}_n)-\mathbf{f}(x,\mathbf{u})]\cdot\mathbf{v}\,dx\right|$$

$$\leqslant C \sum_{i,j=1}^{M}\sum_{k=1}^{K}\prod_{h=1}^{M}\|u^h+\theta(u_n^h-u^h)\|_{|\alpha|}^{\gamma_{ijk}^h}\,\|v^i\|_{|\alpha|}\,\|u_n^j-u^j\|_{|\alpha|}$$

for some constant C and some $\theta=\theta(x)\in(0,1)$. Therefore,

$$\lim_{n\to\infty}\ \sup_{\|v\|_E\leqslant 1}\left|\int_{\mathbf{R}^N}[\mathbf{f}(x,\mathbf{u}_n)-\mathbf{f}(x,\mathbf{u})]\cdot\mathbf{v}\,dx\right|=0.$$

This together with (8) proves that $J\in C^1(E)$ (see [40]).

LEMMA 3. *J is weakly sequentially compact on E.*

Proof. Let $\{\mathbf{u}_n\}$ be a weakly convergent subsequence of a bounded sequence in E, with weak limit $\mathbf{u} \in E$. Use of the mean value theorem, (A_3), and (9) yield, for some $\theta = \theta(x) \in (0, 1)$,

$$\begin{aligned}
&|J(\mathbf{u}_n) - J(\mathbf{u})| \\
&\quad \leqslant \int_{\mathbf{R}^N} |F(x, \mathbf{u}_n) - F(x, \mathbf{u})| \, dx \\
&\quad = \int_{\mathbf{R}^N} \left| \sum_{i=1}^{M} f^i(x, \mathbf{u} + \theta(\mathbf{u}_n - \mathbf{u}))(u_n^i - u^i) \right| dx \\
&\quad \leqslant C \int_{\mathbf{R}^N} g(x) \sum_{i=1}^{M} \sum_{k=1}^{K} |[\mathbf{u} + \theta(\mathbf{u}_n - \mathbf{u})]^{\beta_{ik}} \, |u_n^i - u^i| \, dx \\
&\quad \leqslant C \sum_{i=1}^{M} \sum_{k=1}^{K} \prod_{h=1}^{M} \|u^h + \theta(u_n^h - u^h)\|_{|\beta_{ik}|+1}^{\beta_{ik}^h} \, \|g(u_n^i - u^i)\|_{|\beta_{ik}|+1}.
\end{aligned} \tag{11}$$

Since $|\beta_{ik}| + 1 < 2N/(N-2)$ for each i, k and $g(x) \to 0$ as $|x| \to \infty$, it is known [7, p. 264] that multiplication by g is a compact mapping from $W_0^{1,2}(\mathbf{R}^N)$ into $L^{|\beta_{ik}|+1}(\mathbf{R}^N)$. Thus $\{\mathbf{u}_n\}$ has a subsequence $\{\tilde{\mathbf{u}}_n\}$ such that $\lim_{n \to \infty} \|g(\tilde{u}_n^i - u^i)\|_{|\beta_{ik}|+1} = 0$ for $i = 1, ..., M$, $k = 1, ..., K$. It then follows from (11) that $J(\tilde{\mathbf{u}}_n) \to J(\mathbf{u})$ as $n \to \infty$.

Proof of Theorem 1. We note by (A_1) that $J(\mathbf{u}_0) > 0$ for some $\mathbf{u}_0 \in E$, and so we can choose $d = J(\mathbf{u}_0)$ in (4). Let $\{\mathbf{u}_n\}$ be a minimizing sequence for (4), i.e., $\lim_{n \to \infty} I(\mathbf{u}_n) = I^*$, $J(\mathbf{u}_n) = d$, and $\mathbf{u}_n \geqslant 0$ without loss of generality by (A_1). Since each $a^i(x)$ is bounded above and below by positive constants, I is an equivalent norm on E. Then $\{\mathbf{u}_n\}$ is bounded in E and therefore has a weakly convergent subsequence, also denoted by $\{\mathbf{u}_n\}$, with weak limit $\mathbf{u} \in E$. Lemma 3 shows that this subsequence has a subsequence $\{\mathbf{u}_n\}$ such that $\lim_{n \to \infty} J(\mathbf{u}_n) = J(\mathbf{u}) = d$. The condition $J(\mathbf{u}) = 0$ if any $u^i \equiv 0$, from (A_1), implies that each u^i is nontrivial and nonnegative throughout $\mathbf{R}^N$, $i = 1, ..., M$. It is a standard consequence of the weak convergence of $\mathbf{u}_n$ to $\mathbf{u}$ that $I(\mathbf{u}) = I^*$. Hence, $\mathbf{u}$ solves the variational problem (4).

By the Euler–Lagrange method there exists a Lagrange multiplier $\lambda \in \mathbf{R}$ such that $I'(\mathbf{u}) = \lambda J'(\mathbf{u})$ in E^*, i.e., by (7) and (8),

$$\int_{\mathbf{R}^N} (\nabla \mathbf{u} \cdot \nabla \mathbf{v} + A\mathbf{u} \cdot \mathbf{v}) \, dx = \lambda \int_{\mathbf{R}^N} \mathbf{f}(x, \mathbf{u}) \cdot \mathbf{v} \, dx \tag{12}$$

for all $\mathbf{v} \in E$. With the choice $v^i = \delta_{ij} u^j$, (12) becomes

$$\int_{\mathbf{R}^N} (|\nabla u^i|^2 + a^i(u^i)^2) \, dx = \lambda \int_{\mathbf{R}^N} f^i(x, \mathbf{u}) \, u^i \, dx, \tag{13}$$

for $i=1, \ldots, M$. Since $0 \not\equiv u^i \geqslant 0$ in $\mathbf{R}^N$ and $f^i(x, \mathbf{u})>0$ if $\mathbf{u}>0$, it follows that $\lambda>0$. Equation (12) means that $\mathbf{u}$ is a weak solution of the system (1). By the regularity hypotheses on $\mathbf{a}$ and $\mathbf{f}$, a standard bootstrap argument shows that each $u^i \in C^{2+\eta}_{\mathrm{loc}}(\mathbf{R}^N)$. The strong maximum principle for $-\Delta u^i + a^i u^i \geqslant 0$ in $\mathbf{R}^N$ implies that each $u^i>0$ in $\mathbf{R}^N$.

COROLLARY 4. *The solution* $\mathbf{u}$ *in Theorem* 1 *has the asymptotic behavior* $\lim_{|x| \to \infty} \mathbf{u}(x)=0$ *and* $\lim_{|x| \to \infty} |(\nabla \mathbf{u})(x)|=0$.

Proof. Let $U=\sum_{i=1}^M u^i$. Then U satisfies the equation $-\Delta U=\phi(x)$, where by (A_3)

$$\begin{aligned}|\phi(x)| &= \left|\sum_{i=1}^{M} \left[\lambda f^i(x, \mathbf{u}(x)) - a^i(x)\, u^i(x)\right]\right| \\ &\leqslant Cg(x) \sum_{i,k} [U(x)]^{|\beta_{ik}|} + a^* U(x).\end{aligned}$$

Since $|\beta_{ik}|<(N+2)/(N-2)$ a bootstrap argument as in [30, Lemma 4.4] yields the estimate

$$\|U\|_{C^{1+\eta}(\overline{B_1(x)})} \leqslant C\, \|U\|_{1,2,B_2(x)}$$

for a constant C independent of $x \in \mathbf{R}^N$, where $B_r(x)$ denotes the ball in $\mathbf{R}^N$ of radius r and centre x. Since each $u^i>0$, this implies the conclusion of Corollary 4.

3. EXISTENCE IN THE CASE $\mathbf{a} \geqslant 0$

To obtain an analogue of Theorem 1 when the components of $\mathbf{a}(x)$ are not bounded below by a positive constant, e.g., when some of the components are identically zero, we need to sharpen assumptions (A_2) and (A_3) for $\mathbf{u}>0$ by adjoining the following conditions:

(A_5) There exists a constant $C>0$ such that $u^i f^i(x, \mathbf{u}) \geqslant CF(x, \mathbf{u})$ for all $\mathbf{u} \geqslant 0$, $x \in \mathbf{R}^N$, $i=1, \ldots, M$.

(A_6) These exist continuous functions $g^k\colon \mathbf{R}^N \to \mathbf{R}_+$ with $g^k(x)= 0(|x|^{-b_k})$ as $|x| \to \infty$, $b_k>0$, $k=1, \ldots, K$, such that

$$0 \leqslant f^i(x, \mathbf{u}) \leqslant \sum_{k=1}^{K} g^k(x)\, \mathbf{u}^{\beta_{ik}}, \qquad x \in \mathbf{R}^N, \qquad \mathbf{u} \in \mathbf{R}^M_+$$

for multi-indices β_{ik} satisfying

$$\begin{cases} 1 < |\beta_{ik}| < \dfrac{N+2}{N-2} & \text{if} \quad b_k \geqslant 2 \\ \dfrac{N+2-2b_k}{N-2} < |\beta_{ik}| < \dfrac{N+2}{N-2} & \text{if} \quad 0 < b_k < 2 \end{cases}$$

$i = 1, ..., M$, $k = 1, ..., K$.

Conditions (A_5) and (A_6) together imply the existence of multi-indices α_k such that

$$F(x, \mathbf{u}) \leqslant \frac{1}{C} \sum_{k=1}^{K} g^k(x)\, \mathbf{u}^{\alpha_k}, \qquad x \in \mathbf{R}^N, \qquad \mathbf{u} \in \mathbf{R}^M_+, \tag{14}$$

where

$$\begin{aligned} 2 < |\alpha_k| < \frac{2N}{N-2} \quad & \text{if} \quad b_k \geqslant 2 \\ \frac{2N-2b_k}{N-2} < |\alpha_k| < \frac{2N}{N-2} \quad & \text{if} \quad 0 < b_k < 2 \end{aligned} \tag{15}$$

for $k = 1, ..., K$.

The necessity of the conditions on $|\beta_{ik}|$ in (A_6) will be indicated in Section 5. This, of course, is known in the scalar case $M = 1$ [11, 12, 16, 18, 21, 22, 27, 32].

THEOREM 5. *If* $a^i \geqslant 0$ *in* $\mathbf{R}^N$ *and* (A_1)–(A_6) *hold, the system* (1) *has a positive solution pair* λ, $\mathbf{u}$ *with* $\mathbf{u} \in D \cap C^{2+\eta}_{\text{loc}}(\mathbf{R}^N)$ *such that* $\mathbf{u}(x) = 0(|x|^{(2-N)/2})$ *as* $|x| \to \infty$. *If* a^i *is identically zero for* $i = 1, ..., M$, *this asymptotic estimate can be replaced by* $\mathbf{u}(x) = 0(|x|^{2-N})$ *as* $|x| \to \infty$.

The idea of the proof is to apply Theorem 1 to the sequence of systems

$$-\Delta u^i + \left[a^i(x) + \frac{1}{n}\right] u^i = \lambda f^i(x, \mathbf{u}), \qquad x \in \mathbf{R}^N, \qquad n = 1, 2, ..., \tag*{$(16)_n$}$$

of type (1). We are going to prove, for a positive solution pair λ_n, $\mathbf{u}_n$ of $(16)_n$, that $\{\lambda_n\}$, $\{\mathbf{u}_n\}$ have subsequences that converge to a positive solution pair λ, $\mathbf{u}$ of (1) as $n \to \infty$, with $\mathbf{u} \in D \cap C^{2+\eta}_{\text{loc}}(\mathbf{R}^N)$. If each $a^i \equiv 0$, the estimate $\mathbf{u}(x) = 0(|x|^{2-N})$ as $|x| \to \infty$ will then follow from Theorem 9 (in Section 4). The analogue of (5) for Eq. $(16)_n$ is

$$I_n(\mathbf{u}) = \frac{1}{2} \int_{\mathbf{R}^N} \left[|\nabla \mathbf{u}|^2 + A\mathbf{u} \cdot \mathbf{u} + \frac{1}{n} |\mathbf{u}|^2 \right] dx, \qquad \mathbf{u} \in E, \qquad n = 1, 2, ... \tag{17}$$

LEMMA 6. *Under the assumptions of Theorem* 5, *for every* $n=1, 2, \ldots$, *the system* $(16)_n$ *has a positive solution pair* $\lambda_n, \mathbf{u}_n$, *with* $\mathbf{u}_n \in E \cap C^{2+\eta}_{\mathrm{loc}}(\mathbf{R}^N)$, *such that the sequences* $\{\lambda_n\}$ *and* $\{\| \, |\nabla \mathbf{u}_n| \, \|_2\}$ *are bounded.*

Proof. Theorem 1 guarantees the existence of such a pair $\lambda_n, \mathbf{u}_n$, where $\mathbf{u}_n$ solves the variational problem

$$I_n^* = \inf\{I_n(\mathbf{u}) : \mathbf{u} \in E, J(\mathbf{u}) = d > 0\}. \tag{18}$$

It follows from (17) and (18) that

$$I_n(\mathbf{u}_n) = I_n^* \leqslant \inf\{I_1(\mathbf{u}) : \mathbf{u} \in E, J(\mathbf{u}) = d\} = I_1(\mathbf{u}_1),$$

implying the boundedness of $\{\| \, |\nabla \mathbf{u}_n| \, \|_2\}$. The analogue of (13) for $\lambda_n, \mathbf{u}_n$ is

$$\int_{\mathbf{R}^N} \left[|\nabla u_n^i|^2 + \left[a^i + \frac{1}{n} \right] (u_n^i)^2 \right] dx = \lambda_n \int_{\mathbf{R}^N} f^i(x, \mathbf{u}_n)\, \mathbf{u}_n^i \, dx,$$

$$i = 1, \ldots, M, \qquad n = 1, 2, \ldots. \tag{19}$$

As a consequence of (17)–(19) and (A_5) we obtain

$$2I_1(\mathbf{u}_1) \geqslant 2I_n(\mathbf{u}_1) \geqslant 2I_n(\mathbf{u}_n) > \lambda_n \int_{\mathbf{R}^N} f^i(x, \mathbf{u}_n)\, u_n^i \, dx$$

$$\geqslant C\lambda_n \int_{\mathbf{R}^N} F(x, \mathbf{u}_n)\, dx = C\lambda_n J(\mathbf{u}_n) = C\, d\lambda_n,$$

showing that $\{\lambda_n\}$ is bounded.

LEMMA 7. *The sequence* $\{\mathbf{u}_n(x)\}$ *in Lemma* 6 *is uniformly bounded in* $\mathbf{R}^N$. *Furthermore, there exists a constant* C, *independent of* n, *such that*

$$u_n^i(x) \leqslant C\, |x|^{(2-N)/2}, \qquad |x| \geqslant 1, \qquad i = 1, \ldots, M, \qquad n = 1, 2, \ldots. \tag{20}$$

Proof. From $(16)_n$, (A_6), and the boundedness of $\{\lambda_n\}$ (by Lemma 6), the function defined by $U_n = \sum_{i=1}^M u_n^i$ is a subsolution of the linear equation

$$-\Delta U = C h_n(x)\, U, \qquad x \in \mathbf{R}^N, \qquad n = 1, 2, \ldots \tag{21}$$

for some positive constant C, independent of n, where

$$h_n(x) = \sum_{i=1}^{M} \sum_{k=1}^{K} g^k(x) [U_n(x)]^{|\beta_{ik}| - 1}. \tag{22}$$

For $|x| = 2R \geqslant 1$ and a fixed constant $s > N/2$, we define

$$\phi(R, n) = R^{2-N/s} \, \|h_n\|_{s, B_R(x)}.$$

Trudinger's a priori estimate [39, Theorem 5.1] for subsolutions of (21) implies that

$$\sup_{y \in B_{R/2}(x)} U_n(y) \leqslant C_0 R^{(2-N)/2} \, \|U_n\|_{2N/(N-2), B_R(x)} \tag{23}$$

for a constant C_0 depending only on N, s, and $\phi(R, n)$. Since $\|\nabla U_n\|_2$ is bounded by Lemma 6, also $\|U_n\|_{2N/(N-2)}$ is bounded by a Sobolev embedding theorem. It will be shown below that $\phi(R, n)$ is uniformly bounded for $2R \geqslant 1$. We can then conclude from (23) that there exists a constant C, independent of n, such that

$$U_n(x) \leqslant CR^{(2-N)/2} \qquad \text{for} \quad |x| = 2R \geqslant 1,$$

implying (20).

For $R = 1$, for example, (23) also implies that $U_n(x)$ is locally uniformly bounded, and hence each $u_n^i(x)$ is uniformly bounded in $\mathbf{R}^N$ by (20), $i = 1, ..., M$.

To prove that $\phi(R, n)$ is uniformly bounded for $2R \geqslant 1$, we choose s satisfying

$$\frac{N}{2} < s < \frac{2N}{(N-2)(\beta - 1)}, \qquad \beta = \max_{i,k} |\beta_{ik}|, \tag{24}$$

possible since $(N-2)(\beta - 1) < 4$ by (A_6). Let Hölder exponents p_{ik}, q_{ik} be defined by

$$p_{ik} = \frac{2N}{s(N-2)(|\beta_{ik}| - 1)}, \qquad q_{ik} = \frac{2N}{2N - s(N-2)(|\beta_{ik}| - 1)}. \tag{25}$$

Then $p_{ik} > 1$, $q_{ik} > 1$, and $p_{ik}^{-1} + q_{ik}^{-1} = 1$. Hölder's inequality applied to (22) yields

$$\|h_n\|^s_{s, B_R(x)} \leqslant (MK)^{s-1} \sum_{i,k} \|(g^k)^s\|_{q_{ik}, B_R(x)} \, \|U_n^{s(|\beta_{ik}| - 1)}\|_{p_{ik}, B_R(x)}. \tag{26}$$

Note from (25) that $s(|\beta_{ik}| - 1) \, p_{ik} = 2N/(N-2)$. Since $\|U_n\|_{2N/(N-2)}$ is uniformly bounded, it then follows from (A_6) and (26) that there exists a constant C, independent of n, such that

$$[\phi(R, n)]^s = R^{2s-N} \, \|h_n\|^s_{s, B_R(x)} \leqslant C \sum_{i,k} R^{\sigma_{ik}},$$

where

$$\sigma_{ik} = 2s - N - sb_k + \frac{N}{q_{ik}} = \frac{s}{2}\left[4 - 2b_k - (N-2)(|\beta_{ik}| - 1)\right].$$

Since $\sigma_{ik} < 0$ for all i, k by (A_6), $\phi(R, n)$ is uniformly bounded for $2R \geqslant 1$. This completes the proof of Lemma 7.

Proof of Theorem 5. Let $\{\lambda_n\}$, $\{\mathbf{u}_n\}$ be the sequences in Lemma 6. Since each $\{u_n^i\}$, $i = 1, ..., M$ is uniformly bounded in $\mathbf{R}^N$ by Lemma 7, standard elliptic regularity theory applied to the ith component Eq. (1) shows that $\{u_n(x)\}$ has a subsequence which converges locally uniformly in $C^2(\mathbf{R}^N)$ to a function $u^i \in C^2_{\text{loc}}(\mathbf{R}^N)$ satisfying

$$u^i(x) \leqslant C\,|x|^{(2-N)/2}, \qquad |x| \geqslant 1, \qquad i = 1, ..., M. \tag{27}$$

In view of Lemma 6, $\{\lambda_n\}$ has a convergent subsequence with limit $\lambda \geqslant 0$. We can therefore let $n \to \infty$ in $(16)_n$ to conclude that $\mathbf{u}$ is a nonnegative solution vector of the system (1) whose components satisfy (27).

To prove that each $u^i(x)$ is not identically zero, we first note from (15) that $(N-2)\,|\alpha_k| = 2N - 2b_k + e_k$ for some $e_k > 0$, $k = 1, ..., K$. Exponents for Hölder's inequality will now be selected to be

$$p_k = \frac{2N + e_k}{(N-2)\,|\alpha_k|}, \qquad q_k = \frac{2N + e_k}{2N + e_k - (N-2)\,|\alpha_k|}. \tag{28}$$

Then $b_k q_k > N$, $p_k > 1$, $q_k > 1$, and $p_k^{-1} + q_k^{-1} = 1$. Use of (6), (14), (18), and (A_6) shows that there exists a positive constant C, independent of n, such that

$$0 < d = J(\mathbf{u}_n) \leqslant C \int_{\mathbf{R}^N} \sum_{k=1}^{K} \left[1 + |x|^{b_k}\right]^{-1} \left[\mathbf{u}_n(x)\right]^{\alpha_k} dx. \tag{29}$$

Let δ_k be a multi-index with the properties $0 < \delta_k^i < \alpha_k^i$ for $i = 1, ..., M$ and $|\delta_k| = e_k/(N-2)\,p_k$, $k = 1, ..., K$. This is possible since

$$|\alpha_k| - |\delta_k| = \frac{2N}{(N-2)\,p_k} > 0, \qquad k = 1, ..., K.$$

We define functions ϕ_{nk} and ψ_{nk} in $\mathbf{R}^N$ by

$$\phi_{nk}(x) = \left[\mathbf{u}_n(x)\right]^{\alpha_k - \delta_k}, \qquad \psi_{nk}(x) = \left[1 + |x|^{b_k}\right]^{-1} \left[\mathbf{u}_n(x)\right]^{\delta_k},$$

$k = 1, ..., K$; $n = 1, 2, ...$ Hölder's inequality applied to (29) yields

$$0 < d \leqslant \tilde{C} \sum_k \|\phi_{nk}\|_{p_k} \|\psi_{nk}\|_{q_k} \tag{30}$$

for some constant $\tilde{C}$, independent of n. By (9),

$$\|\phi_{nk}\|_{p_k}^{p_k} \leqslant \prod_{h=1}^{M} \|u_n^h\|_{p_k|\alpha_k - \delta_k|}^{p_k(\alpha_k^h - \delta_k^h)},$$

and hence it follows from $p_k |\alpha_k - \delta_k| = 2N/(N-2)$ that $\{\|\phi_{nk}\|_{p_k}\}$ is a bounded sequence, $k = 1, ..., K$. Since $\{u_n(x)\}$ is uniformly bounded by Lemma 7 and $b_k q_k > N$, also $\{\|\psi_{nk}\|_{q_k}\}$ is bounded. From the pointwise convergence of $\psi_{nk}(x)$ to the function

$$\psi_k(x) = [1 + |x|^{b_k}]^{-1} [u(x)]^{\delta_k}$$

as $n \to \infty$, it follows that $\psi_k^{q_k} \in L^1(\mathbf{R}^N)$. Then the dominated convergence theorem applied to (30) gives, for some positive constant C,

$$0 < d \leqslant C \sum_{k=1}^{K} \|\psi_k\|_{q_k},$$

which is impossible if $u^i(x)$ is identically zero for any integer $i = 1, ..., M$.

Since $(\nabla u_n^i)(x) \to (\nabla u^i)(x)$ pointwise in $\mathbf{R}^N$ and $\{\|\nabla u_n^i\|_2\}$ is bounded, Fatou's lemma implies that $u^i \in D_0^{1,2}(\mathbf{R}^N)$ for $i = 1, ..., M$, i.e., $\mathbf{u} \in D$. If $\lambda = 0$, $u^i(x)$ would be a nontrivial nonnegative solution of $\Delta u^i = 0$ in $\mathbf{R}^N$ with uniform limit zero as $|x| \to \infty$ by (27), contrary to Liouville's theorem, and hence $\lambda > 0$. Then the strong maximum principle for $\Delta u^i \leqslant 0$ shows that $u^i(x) > 0$ throughout $\mathbf{R}^N$, $i = 1, ..., M$. If each $a^i \equiv 0$, Theorem 9 (in Section 4) implies that each $u^i(x) = 0(|x|^{2-N})$ as $|x| \to \infty$. This completes the proof of Theorem 5.

4. A Priori Asymptotic Estimates

This section concerns the asymptotic behavior of any nontrivial nonnegative solution $\mathbf{u}$ of (1) such that each component $u^i \in D_0^{1,2}(\mathbf{R}^N)$. Similar estimates for the case $M = 1$ were obtained recently by Egnell [12] via a different procedure.

Theorem 8. *Suppose f^i satisfies* (A_3) *as well as the conditions stated below* (1), $i = 1, ..., M$. *If $\mathbf{u} \in D$ is a nonnegative solution of* (1), *then $\mathbf{u}$ is bounded, $\mathbf{u} \in C^2_{\mathrm{loc}}(\mathbf{R}^N)$, and*

$$\lim_{|x| \to \infty} \mathbf{u}(x) = \lim_{|x| \to \infty} |\nabla \mathbf{u}(x)| = 0. \tag{31}$$

Proof. The function $U = \sum_{i=1}^{M} u^i$ (assumed to be nontrivial) satisfies the linear equation $-\Delta U = H(x)\, U$ in $\mathbf{R}^N$, where by (A_3)

$$H(x) \leqslant C \sum_{i=1}^{M} \sum_{k=1}^{K} [U(x)]^{|\beta_{ik}|-1}, \tag{32}$$

similarly to (22). Since $U \in L^{2N/(N-2)}(\mathbf{R}^N)$ from the embedding $D_0^{1,2}(\mathbf{R}^N) \hookrightarrow L^{2N/(N-2)}(\mathbf{R}^N)$ and since $|\beta_{ik}| - 1 < 4/(N-2)$, Hölder's inequality with exponents

$$p_k = \frac{4}{(N-2)(|\beta_{ik}|-1)}, \qquad q_k = \frac{4}{4-(N-2)(|\beta_{ik}|-1)}$$

shows that $\|H\|_{N/2, B_r(x)} \to 0$ as $r \to 0$ uniformly in $\mathbf{R}^N$. Results of Brézis and Kato [8] then imply that $U \in L^q(\mathbf{R}^N)$ for all $q \geqslant 2N/(N-2)$, and hence the norms

$$\|U\|_{q, B_2(x)}, \ \|G\|_{q/\beta, B_2(x)} \tag{33}$$

are bounded functions of $x \in \mathbf{R}^N$ and have limits zero as $|x| \to \infty$, where $G = HU$ and $\beta = \min_{i,k} |\beta_{ik}|$. Standard a priori interior estimates [17, Chap. 8] for $-\Delta U = G$ show, if q is sufficiently large in (33), that $U(x)$ is bounded in $\mathbf{R}^N$ and $U \in C^2_{\mathrm{loc}}(\mathbf{R}^N)$. Then $\|U\|_{2, B_2(x)}$, as well as the norms (33), has limit zero as $|x| \to \infty$ (by Hölder's inequality), and consequently interior Hölder estimates for $-\Delta U = G$ [17, Theorems 8.24 and 8.32] imply that

$$\lim_{x \to \infty} \|U\|_{C^{1+\eta}(\overline{B_1(x)})} = 0.$$

Since each $u^i \geqslant 0$ this proves (31).

The next theorem concerns the a priori asymptotic behavior at ∞ of solutions $\mathbf{u}$ of systems of type

$$-\Delta u^i = f^i(x, \mathbf{u}), \qquad x \in \Omega, \qquad i = 1, ..., M \tag{34}$$

in exterior domains $\Omega \subset \mathbf{R}^N$, which for our purpose can be taken as $\Omega = \{x \in \mathbf{R}^N : |x| > R\}$, $R > 0$. We assume that each $f^i \in C^\eta_{\mathrm{loc}}(\Omega \times \mathbf{R}^M, \mathbf{R}_+)$, $0 < \eta < 1$, and that f^i satisfies the growth condition (A_6) (see Section 3) in Ω, i.e., $\mathbf{R}^N$ in (A_6) is replaced by Ω. The notation $\mathbf{u} \in D(\Omega)$ is defined to mean that $h\mathbf{u} \in D$ for some smooth nonnegative function h with $h(x) = 1$ for all sufficiently large x.

THEOREM 9. *Under the above conditions on* $\mathbf{f}$, *every positive solution* $\mathbf{u} \in D(\Omega)$ *of* (34) *is minimal, i.e.,* $|x|^{N-2} u^i(x)$ *is bounded above and below by positive constants in* Ω, $i = 1, ..., M$.

Proof. Kelvin's transformation

$$y = x/|x|^2, \qquad v^i(y) = |x|^{N-2} u^i(x), \qquad i = 1, ..., M \tag{35}$$

maps (34) into

$$-\Delta v^i = \frac{1}{|y|^{N+2}} f^i\left(\frac{y}{|y|^2}, |y|^{N-2}\mathbf{v}(y)\right), \qquad y \in \Omega', \qquad i = 1, ..., M,$$

where $\Omega' = \{y \in \mathbf{R}^N : 0 < |y| < 1/R\}$. Then v^i can be regarded as a solution of the linear equation

$$-\Delta v^i = H^i(y)\, v^i, \qquad y \in \Omega', \tag{36}$$

where

$$H^i(y) = |y|^{-N-2} [v^i(y)]^{-1} f^i\left(\frac{y}{|y|^2}, |y|^{N-2}\mathbf{v}(y)\right).$$

It follows from assumption (A_6) that

$$H^i(y) \leqslant \sum_{k=1}^{K} H^{ik}(y), \qquad y \in \Omega',$$

where

$$H^{ik}(y) = |y|^{\rho_{ik}} g^k\left(\frac{y}{|y|^2}\right) (v^1(y))^{\beta^1_{ik}} \cdots (v^i(y))^{\beta^i_{ik}-1} \cdots (v^M(y))^{\beta^M_{ik}} \tag{37}$$

and $\rho_{ik} = |\beta_{ik}|\,(N-2) - N - 2$, $i = 1, ..., M$, $k = 1, ..., K$. Let $V(y) = \sum_{i=1}^{M} v^i(y)$. Then (37) and (A_6) show that a positive constant C exists such that

$$H^{ik}(y) \leqslant C\,|y|^{\rho_{ik}+b_k} [V(y)]^{|\beta_{ik}|-1}, \qquad y \in \Omega'. \tag{38}$$

If we can show that $H^{ik} \in L^s(\Omega')$ for some $s > N/2$, $i = 1, ..., M$, $k = 1, ..., K$, then $H^i \in L^s(\Omega')$, and Serrin's theorem on isolated singularities [35, p. 220] applied to (36) near $y = 0$ shows that either $v^i(y)$ or $|y|^{N-2} v^i(y)$ is bounded above and below by positive constants in a deleted neighbourhood of $y = 0$. Thus either $|x|^{N-2} u^i(x)$ or $u^i(x)$ is bounded above and below by positive constants in Ω. However, the second possibility is precluded by the assumption that $\mathbf{u} \in D(\Omega)$, proving the minimality of $\mathbf{u}$.

To show that $H^{ik} \in L^s(\Omega')$ for $s = N/(2-\varepsilon)$, we fix $\varepsilon > 0$ satisfying

$$4 - 2b_k + 2\varepsilon < (N-2)(|\beta_{ik}| - 1) < 4 - 2\varepsilon,$$

which is possible by (A_6). This is equivalent to

$$8 - 2b_k - 2N/s < (N-2)(|\beta_{ik}| - 1) < 2N/s. \tag{39}$$

Hölder's inequality with exponents

$$p_{ik}=\frac{2N}{s(N-2)(|\beta_{ik}|-1)},\qquad q_{ik}=\frac{2N}{2N-s(N-2)(|\beta_{ik}|-1)}$$

applied to (38) gives

$$\|H^{ik}\|^s_{s,\Omega'}\leqslant \|\,|y|^{s(\rho_{ik}+b_k)}\|_{q_{ik},\Omega'}\,\|V\|^{s(|\beta_{ik}|-1)}_{2N/(N-2),\Omega'}. \tag{40}$$

It is well known and easily verified that the assumption $\mathbf{u}\in D(\Omega)$ implies that $h'\mathbf{v}\in D$ for some smooth nonnegative function h' with $h'(y)=1$ in a neighborhood of the origin, from which $V\in L^{2N/(N-2)}(\Omega')$ by Sobolev embedding. Since the left inequality (39) is equivalent to

$$s(\rho_{ik}+b_k)\,q_{ik}>-N$$

by a routine computation, the conclusion $H^{ik}\in L^s(\Omega')$ is a consequence of (40). This completes the proof of Theorem 9.

Remarks. If a^i is identically zero for each $i=1,\ldots,M$, Theorem 9 can be applied to the solution $\mathbf{u}(x)$ of (1) obtained in Theorem 5, as already mentioned in Section 3. Elliptic regularity theory for (36) enables us to prove that the function $v^i(y)$ in (35) actually has a positive limit as $|y|\to 0$, from which $|x|^{N-2}\,u^i(x)$ has a positive finite limit as $|x|\to\infty$ for $i=1,\ldots,M$; see, e.g., [12].

5. Necessary Conditions for Positive Solutions

Application of the divergence theorem to $(x\cdot\nabla u^i)\,\nabla u^i$ in a ball of radius R, summing over i, and then letting $R\to\infty$, we find that every solution $\mathbf{u}\in E$ of (1) satisfies the Pohožaev-type identity

$$\int_{\mathbf{R}^N} A\mathbf{u}\cdot\mathbf{u}\,dx=\lambda\int_{\mathbf{R}^N}\left[NF(x,\mathbf{u})-\frac{1}{2}(N-2)\,\mathbf{u}\cdot\mathbf{f}(x,\mathbf{u})+x\cdot\nabla_x F(x,\mathbf{u})\right]dx. \tag{41}$$

The notation ∇_x means that the components of $\mathbf{u}$ are held constant during the differentiation. If $a^i\equiv 0$ for each $i=1,\ldots,M$, then (41) holds for every solution $\mathbf{u}\in D$ of (34). Similar Pohožaev identities are used in [6, 11, 13, 26, 30, 36], for example.

If $\mathbf{u}\in E$ is a positive solution of (1) with the prototype potential (3), i.e.,

$$f^i(x,\mathbf{u})=\sum_{k=1}^{K}\alpha^i_k\,g^k(x)(u^1)^{\alpha^1_k}\cdots(u^i)^{\alpha^i_k-1}\cdots(u^M)^{\alpha^M_k},$$

then (41) implies that

$$0 \leqslant \int_{\mathbf{R}^N} \sum_{k=1}^{K} \left[\left(N - \frac{1}{2}(N-2)\,|\alpha_k| \right) g^k(x) + x \cdot \nabla g^k(x) \right] u^{\alpha_k}\, dx. \quad (42)$$

If $g^k(x) > 0$ and $0 \not\equiv x \cdot \nabla g^k(x) \leqslant 0$ throughout $\mathbf{R}^N$ for $k = 1, ..., K$, then necessarily $|\alpha_k| < 2N/(N-2)$ for at least one integer k.

If $|\alpha_k| = 2N/(N-2)$ for all $k = 1, ..., K$, it also follows from (42) that $x \cdot \nabla g^k(x)$ must be positive on a set of sufficiently large measure, for some k, in order that a positive $\mathbf{u} \in E$ (or $\mathbf{u} \in D$ if each $a^i \equiv 0$) of (1) can exist.

We now derive necessary conditions for the existence of a positive solution $\mathbf{u} \in D(\Omega)$ of (34) in an exterior domain Ω. The conditions on $\mathbf{f}$ stated below (34) will be retained, and we also need *lower* bounds

$$f^i(x, \mathbf{u}) \geqslant g^i_*(|x|)\, \mathbf{u}^{\beta_i}, \qquad x \in \Omega, \qquad 0 < \mathbf{u} \leqslant 1, \qquad i = 1, ..., M \quad (43)$$

for positive continuous functions $g^i_* : \mathbf{R}_+ \to \mathbf{R}_+$.

THEOREM 10. *Suppose f^i satisfies conditions* (A_3) *and* (43), *and* $\beta^i_i > 1$. *Then necessary conditions for the existence of a positive solution* $\mathbf{u} \in D(\Omega)$ *of the system* (34) *are*

$$\int^{\infty} g^i_*(r)\, r^{N-1-|\beta_i|(N-2)}\, dr < \infty, \qquad i = 1, ..., M. \quad (44)$$

Proof. By Theorem 8, $u^i(x) \to 0$ as $|x| \to \infty$, and hence the maximum principle for $-\Delta u^i \geqslant 0$ implies that there exist positive constants C and R_0 such that $u^i(x) \geqslant C\,|x|^{2-N}$ for all $|x| > R_0$, $i = 1, ..., M$. It then follows from (34) and (43) that

$$-\Delta u^i \geqslant C^{M-1} p^i(|x|)(u^i)^{\beta^i_i}, \qquad |x| \geqslant R_0, \qquad i = 1, ..., M, \quad (45)$$

where

$$p^i(r) = g^i_*(r)\, r^{-(|\beta_i| - \beta^i_i)(N-2)}.$$

Since $u^i > 0$, a well known oscillation criterion for (45) [29] implies that

$$\int^{\infty} p^i(r)\, r^{N-1-\beta^i_i(N-2)}\, dr < \infty,$$

which is equivalent to (44). (This oscillation criterion has been rediscovered by many authors).

In particular, if (43) is specialized to

$$f^i(x, \mathbf{u}) \geqslant C^i\,|x|^{-b_i}\, \mathbf{u}^{\beta_i}, \qquad x \in \Omega, \qquad \mathbf{u} > 0, \qquad i = 1, ..., M,$$

for some constants b_i and $C^i > 0$, Theorem 10 shows that no positive solution $\mathbf{u} \in D(\Omega)$ of (34) can exist if $|\beta_i| \leqslant (N - b_i)/(N-2)$ for some integer i.

The proof of Theorem 10 also establishes, without condition (A_3), that (44) is a necessary condition for the existence of a minimal positive solution of (34) in Ω, i.e., a solution $\mathbf{u}$ such that $|x|^{N-2}\,\mathbf{u}(x)$ is bounded above and below by positive constants in Ω.

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